IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 16, Issue 2 Ser. III (Mar. – Apr. 2020), PP 44-49
www.iosrjournals.org

On the Basics of Linear Diffusion

Henrik Stenlund *

Abstract: *This study handles the three-dimensional linear diffusion in a new way. A general solution is given without particular initial conditions. In addition, solutions are obtained for a source-sink as a constant in time but spatially varying in three dimensions and having an arbitrary time dependence. An auxiliary function for diffusion is given having an interesting relationship with the concentration. It appears that both the time derivative and the Laplacian of the concentration obey the diffusion equation. An integro-differential equation for diffusion is presented. 1*

Mathematical Classification
Mathematics Subject Classification: 34A30,34A37,34A12
Keywords: diffusion, linear diffusion, thermal conductivity
** The author is grateful to Visilab Signal Technologies for supporting this work.*
*1 Visilab Report **#2020-04***

Date of Submission: 16-04-2020 Date of Acceptance: 01-05-2020

I. Introduction

Diffusion is one of the most important natural phenomena having a close relation with thermal conduction problems. The primary target is here in initial value problems, instead of boundary value problems but no specific initial value problems are treated here. In three dimensions the diffusion equation becomes awkward to solve and usually leads to numerical analysis. In this paper new formulas are given for solving diffusion and thermal conductivity problems in three dimensions. Also other observations are made of the diffusion equation. The well-known work made by pioneers, like Crank [1] and Churchill [2] are classic. A recent article by the author gave a general solution to the non-linear diffusion [3]. That will be used as a starting point here.

In the following is the nonlinear problem and its solution shortly presented. It is applied to the linear problem with no concentration dependence on the diffusion coefficient. A universal solution is given for general three-dimensional problems. The case of general source and sink is solved analogously, first for a time-independent case and then for a case with an arbitrary time dependence. As an observation, an auxiliary function for diffusion is treated next.

II. The General Solution of the Diffusion Equation

2.1 Nonlinear Solution Presented

In an earlier paper [3] was given the general solution for non-linear diffusion having concentration dependence in the diffusion coefficient. One will start from that and solve the linear case in the following. Here c is the concentration. The non-linear diffusion equation in three dimensions will be

$$\frac{\partial c}{\partial t} = \nabla \cdot (D(c)\nabla c) \qquad (1)$$

where (t, **r**) are the time and spatial coordinate vector. No sources nor sinks are present at this point. The simple transformation is used

$$F(c) = \int_{c_0}^{c} D(s)ds \qquad (2)$$

with c0 a constant and D(c) the non-linear diffusion coefficient. It will help in progressing to

$$\nabla \cdot (D(c)\nabla c) = \nabla^2 F(c) \qquad (3)$$

and to the general solution after integrating in time

$$c(\bar{r},t) = c(\bar{r},t=0) + \sum_{n=0} \frac{t^{n+1}}{(n+1)!}\nabla^2(\frac{\partial^n F(c(\bar{r},t))}{\partial t^n})_{t=0} \qquad (4)$$

Here it is assumed the c(**r**, t = 0) and all its derivatives to be known at t = 0.

2.2 Linear Case

One can apply equation (4) to the linear case, a constant D0. A similar approach based on operator formalism is given in [4] and [5]. Since

$$F(c) = D_0(c - c_0) \qquad (5)$$

the first derivative will be

$$\frac{\partial F(c)}{\partial t} = D_0^2 \nabla^2 c \qquad (6)$$

The higher derivatives are straightforward to calculate and the general expression will be

$$\frac{\partial^n F(c)}{\partial t^n} = D_0^{n+1}(\nabla^2)^n c(\bar{r},t) \qquad (7)$$

Therefore one will obtain after a change of index

$$c(\bar{r},t) = \sum_{n=0} \frac{t^n}{n!} D_0^n (\nabla^2)^n c(\bar{r},t=0) \qquad (8)$$

This can be identified as the definition of the time translation operator as a series expansion

$$c(\bar{r},t) = e^{tD_0\nabla^2} c(\bar{r},t=0) \qquad (9)$$

This can actually be transformed to an integral

$$c(\bar{r},t) = \frac{1}{(4\pi D_0 t)^{\frac{3}{2}}} \int d\bar{r'} e^{\frac{-|\bar{r}-\bar{r'}|^2}{4D_0 t}} c(\bar{r'},t=0) \qquad (10)$$

Equation (10) is the universal solution of the linear case in three dimensions when the initial value function for the concentration is known. There are no sources nor any sinks. The integral is taken over the whole space to infinity.

2.3 Time-Independent Sinks And Sources

One can apply (4) to the case of a constant D0 with a general source-sink Q which is known entirely. It is defined in the whole space but can be zero and does not need to be continuous. It must be constant in time.

$$\frac{\partial c}{\partial t} = D_0\nabla^2 c + Q(\bar{r}) \qquad (11)$$

Again

$$F(c) = D_0(c - c_0) \qquad (12)$$

the first derivative will be

$$\frac{\partial F}{\partial t} = D_0^2\nabla^2 c + D_0 Q \qquad (13)$$

The higher derivatives come out as

$$\frac{\partial^n F}{\partial t^n} = D_0^{n+1}(\nabla^2)^n c(\bar{r},t) + D_0^n(\nabla^2)^{n-1}Q(\bar{r}) \qquad (14)$$

Therefore one will obtain after a change of index

$$c(\bar{r},t) = \sum_{n=0} \frac{t^n}{n!} D_0^n(\nabla^2)^n c(\bar{r},t=0) + \sum_{n=0} \frac{t^{n+1}}{(n+1)!} D_0^n(\nabla^2)^n Q(\bar{r}) \qquad (15)$$

The term tQ is absorbed by the latter sum. One can identify the first term processed in the previous case as

$$c_1(\bar{r},t) = e^{tD_0\nabla^2} c(\bar{r},t=0) \qquad (16)$$

leading to

$$c_1(\bar{r},t) = \frac{1}{(4\pi D_0 t)^{\frac{3}{2}}} \int d\bar{r'} e^{\frac{-|\bar{r}-\bar{r'}|^2}{4D_0 t}} c(\bar{r'},t=0) \qquad (17)$$

The second term in equation (15) is processed accordingly and is recognized as the time integral

$$c_2(\bar{r},t) = \int_0^t dt \frac{1}{(4\pi D_0 t)^{\frac{3}{2}}} \int d\bar{r'} e^{\frac{-|\bar{r}-\bar{r'}|^2}{4D_0 t}} Q(\bar{r'}) \qquad (18)$$

The complete solution is therefore

$$c(\bar{r},t) = \frac{1}{(4\pi D_0 t)^{\frac{3}{2}}} \int d\bar{r'} e^{\frac{-|\bar{r}-\bar{r'}|^2}{4D_0 t}} c(\bar{r'}, t=0) + \tag{19}$$

$$+ \int_0^t dt' \frac{1}{(4\pi D_0 t')^{\frac{3}{2}}} \int d\bar{r'} e^{\frac{-|\bar{r}-\bar{r'}|^2}{4D_0 t'}} Q(\bar{r'}) \tag{20}$$

Equation (20) is the universal solution of the linear diffusion in three dimensions when the initial value function for the concentration is known with a source-sink, constant in time. The first term is the complementary function and the second term is the particular integral.

2.4 Time-Dependent Sinks And Sources

One can solve equation (11) in the case of a constant D0 with a general source and sink Q which is known entirely and having an arbitrary time dependence. It is defined in the whole space but can be zero and does not need to be continuous. This can not be solved in the same manner as in the previous case as the time dependence of the Q is arbitrary. One must use a completely different approach. One recognizes that the equation

$$\frac{\partial c(\bar{r},t)}{\partial t} = D_0 \nabla^2 c(\bar{r},t) + Q(\bar{r,t}) \tag{21}$$

is of first order in time. It has an operator constant in time to be treated as a constant but maintaining the operator order. Standard methods give the solution in operator form

$$c(\bar{r},t) = e^{tD_0\nabla^2} c(\bar{r},0) + \int_0^t dt e^{-(t'-t)D_0\nabla^2} Q(\bar{r},t') \tag{22}$$

or opened up as integrals

$$c(\bar{r},t) = \frac{1}{(4\pi D_0 t)^{\frac{3}{2}}} \int d\bar{r'} e^{\frac{-|\bar{r}-\bar{r'}|^2}{4D_0 t}} c(\bar{r'}, t=0) + \tag{23}$$

$$+ \int_0^t dt' \frac{1}{(4\pi D_0 (t-t'))^{\frac{3}{2}}} \int d\bar{r'} e^{\frac{-|\bar{r}-\bar{r'}|^2}{4D_0 (t-t')}} Q(\bar{r'}, t') \tag{24}$$

Equation (24) is the universal solution of the linear diffusion in three dimensions when the initial value function for the concentration is known. It has a source-sink with an arbitrary time dependence. Again, the first term is the complementary function and the second term is the particular integral.

III. The Auxiliary Function For The Linear Diffusion Equation

In

$$\frac{\partial c}{\partial t} = D_0 \nabla^2 c \tag{25}$$

one can take into use the following auxiliary function recalling the property of the delta function in three dimensions.

$$c(\bar{r},t) = -\frac{1}{4\pi}\int \frac{d\bar{r'}\rho(\bar{r'},t)}{|\bar{r}-\bar{r'}|} \qquad (26)$$

This will lead to

$$\nabla^2 c = \rho = \frac{1}{D_0}\frac{\partial c}{\partial t} \qquad (27)$$

This means that **rho** is actually both the time derivative of the concentration and the Laplacian of the concentration. On the other hand, by substituting equation (26) to (25) will give

$$D_0\rho = -\frac{1}{4\pi}\frac{\partial}{\partial t}\int \frac{d\bar{r'}\rho(\bar{r'},t)}{|\bar{r}-\bar{r'}|} \qquad (28)$$

Applying the Laplacian to this will give

$$\frac{\partial \rho}{\partial t} = D_0\nabla^2\rho \qquad (29)$$

The auxiliary function **rho** obeys the diffusion equation but with different initial conditions. It is straightforward to verify that also the time integral of the concentration obeys the same diffusion equation, as follows. One may consider another diffusion as

$$\frac{\partial \kappa}{\partial t} = D_0\nabla^2\kappa \qquad (30)$$

and suggest the auxiliary function c

$$\kappa(\bar{r},t) = -\frac{1}{4\pi}\int \frac{d\bar{r'}c(\bar{r'},t)}{|\bar{r}-\bar{r'}|} \qquad (31)$$

to it. One will get

$$\frac{\partial c}{\partial t} = D_0\nabla^2 c \qquad (32)$$

Therefore it goes back to

$$\nabla^2 \kappa = c = \frac{1}{D_0}\frac{\partial \kappa}{\partial t} \qquad (33)$$

Kappa is the time integral of c and obeys the diffusion equation. At this point it is obvious that both an Nth time derivative and an Nth power of a Laplacian of the concentration also obeys the diffusion equation. The same applies to any multiple of time integrals of the original concentration and volume integrals like equation (31). One can get the following equation as in (28)

$$D_0 c = -\frac{1}{4\pi}\frac{\partial}{\partial t}\int \frac{d\bar{r'}\, c(\bar{r'}, t)}{|\bar{r} - \bar{r'}|} \qquad (34)$$

This is an integro-differential equation for diffusion. It is equivalent to the original diffusion equation and can be verified for instance by substituting it inside the integral. One can apply the Green's II theorem for integrals and get rid of the Laplacian and getting a Dirac delta function instead. That will deliver the c(**r**; t) itself. The surface integral term generated by the Green's II theorem will disappear since the boundaries are in the infinity and the concentration is supposed to vanish there. Obviously, there exist other shorter proofs, like subjecting the equation above to a Laplacian.

IV. Conclusions

Equations (10), (20) and (24) appear to be new three-dimensional extensions of the well-known one-dimensional general solutions. They will serve as starting points both for analytical investigations and numerical work with various initial conditions. The general source-sink added to the diffusion equation with no time dependence is solved in a similar manner, supposing the term to be completely known. If there is any time dependence in it, this method will break down as it is based on initial values only. The temporal behavior of an arbitrary function can not be controlled by one point in time. An example would be a human- controlled source being zero at the initial time acting as a driving force of the diffusion equation. That case is solved by using a standard method for linear differential equations and using differential operators treated first as constants and which then can be transformed to integrals.

The auxiliary function □ in equation (27) for diffusion connects both the time derivative and the Laplacian of the concentration. It obeys the diffusion equation with different initial conditions. It appears that even the Nth time derivative and the Nth time integral of the concentration obey the diffusion equation. The same applies to the Nth power of the Laplacian applied to the concentration. The equation (34) is a simple integro-differential equation equivalent to the regular differential equation involving both the time derivative and the Laplacian.

References

[1]. Crank, John: *Mathematics of Diffusion*, Clarendon Press(1979)
[2]. Churchill, Ruel: *Operational Mathematics*, McGraw-Hill Kogakusha, 3rd edition (1972), Tokyo
[3]. Stenlund, H.: *On the Basics of the Non-Linear Diffusion*, arXiv:2003.06282v1 [math.GM] 12 Mar 2020.
[4]. Stenlund, H:*On Solving the Cauchy Problem with Propagators*, arXiv:1411.1402v1[math.AP], 5th November 2014
[5]. Stenlund, H:A *Closed-Form Solution to the Arbitrary Order Cauchy Problem with Propagators*, arXiv:1411.6890v1 [math.GM] 24th November 2014

Henrik Stenlund. " On the Basics of Linear Diffusion. " *IOSR Journal of Mathematics (IOSR-JM)*, 16(2), (2020): pp. 44-49.