

Inversion Formula

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July 27, 2010

Abstract

This work introduces a new inversion formula for analytical functions. It is simple, generally applicable and straightforward to use both in hand calculations and for symbolic machine processing. It is easier to apply than the traditional Lagrange-Bürmann formula since no taking limits is required. This formula is important for inverting functions in physical and mathematical problems. The remainder term has been evaluated both for real and complex variable cases to give an error estimate for a broken series.¹

0.1 Keywords

inversion of functions, Taylor series, Lagrange-Bürmann inversion formula, reversion of series

0.2 Mathematical Classification

Mathematics Subject Classification 2010: 11A25, 40E99, 32H02

1 Introduction

1.1 General

The inversion of an analytic function $f(z)$ with $z, u \in C$

$$f(z) = u \tag{1}$$

is defined as

$$z = g(u) \tag{2}$$

*The author is grateful to Visilab Signal Technologies for supporting this work.

¹Visilab Report #2010-07. Revision 1 written in L^AT_EX with some sentences corrected, 2-1-2011. Revision 2 with equation (17) typo fixed. Revision 3 Dec. 2017 with derivation of the remainder term

There is no general simple method known to determine $g(u)$ unless the variable z can be readily solved from $f(z)$. Lagrange [1] was the first to find a useful series expansion. Bürmann [2] and [8] generalized it to the Lagrange-Bürmann formula.

Good [3] extended the Lagrange-Bürmann formula to multiple variables. His formula is known as the Lagrange-Good formula and Hofbauer [4] supplied the proof. A number of investigations has been published over the Lagrange-Bürmann formula for various applications, like Zhao [5] and Merlini et al. [6]. Sokal [7] recently introduced a new generalization of the Lagrange-Bürmann formula. We first express the Lagrange-Bürmann inversion formula which is the present standard method for calculating the inverse. The new inversion formula is derived next.

1.2 The Lagrange-Bürmann Inversion Formula

Lagrange [1] and Bürmann [2] introduced an inversion formula for a function $f(z)$ of a complex variable z .

$$f(z) = u \tag{3}$$

with f being analytic at some point z_0 and the first derivative at z_0 , is required to be nonzero.

$$\left[\frac{df(z)}{dz}\right]_{z_0} \neq 0 \tag{4}$$

$f(z)$ has a value u_0 at z_0 . The inverse function is $g(u)$

$$z = g(u) = g(f(z)) \tag{5}$$

The Lagrange inversion formula or the Lagrange-Bürmann formula is a Taylor series as follows.

$$z = z_0 + \sum_{n=1}^{\infty} \frac{(u - u_0)^n}{n!} \left[\lim_{z \rightarrow z_0} \left[\frac{d^{n-1}}{dz^{n-1}} \left(\frac{z - z_0}{f(z) - u_0} \right)^n \right] \right] \tag{6}$$

Proof of this formula can be found in [1] and [2]. Taking limits in terms in equation (6) usually requires lengthy calculations and a repeated use of L'Hospital's rule to get rid of the singularity. All terms belonging to a certain coefficient need to be kept together to determine the limit properly. This may be a very laborious task in hand calculations.

2 The Inversion Formula

Using the annotation of the preceding chapter, let

$$u = f(z) \quad z, u \in C \tag{7}$$

and $f(z)$ be analytic over the interior of a circle

$$r = |z - z_0| \tag{8}$$

Let the inverse function $g(u)$ be analytic over the interior of a circle R_0 at u_0

$$R_0 = |u - u_0| \quad (9)$$

We have a Taylor series

$$z = z_0 + \sum_{n=1}^{\infty} \frac{(u - u_0)^n}{n!} \left[\frac{d^n}{du^n} g(u) \right]_{u_0} \quad (10)$$

This series converges over the circle R_1 (as in equation (9)). The equation (10) is very difficult to be used any further as such. Higher derivatives of $g(u)$ are requested and to get them, one would need the $g(u)$. We can use derivatives of $f(z)$ instead of $g(u)$. In order to circumvent the generation of progressively complicated terms, we proceed as follows. Differentiate equation (11) below.

$$z = g(u) \quad (11)$$

to obtain

$$\frac{d}{dz} z = 1 = \left(\frac{d}{du} g \right) \left(\frac{d}{dz} u \right) = \left(\frac{d}{du} g(u) \right) \left(\frac{d}{dz} f(z) \right) \quad (12)$$

and solve it as

$$\frac{d}{du} g(u) = \frac{1}{\left(\frac{d}{dz} f(z) \right)} \quad (13)$$

Differentiate (13) further and solve it for

$$\frac{d^2}{du^2} g(u) = \frac{1}{\left(\frac{d}{dz} f(z) \right)} \cdot \left[\frac{d}{dz} \frac{1}{\left(\frac{d}{dz} f(z) \right)} \right] \quad (14)$$

In the same manner the n 'th derivative would be solved as

$$\frac{d^n}{du^n} g(u) = \frac{1}{\left(\frac{d}{dz} f(z) \right)} \cdot \left[\frac{d}{dz} \frac{1}{\left(\frac{d}{dz} f(z) \right)} \cdot \left[\frac{d}{dz} \frac{1}{\left(\frac{d}{dz} f(z) \right)} \cdots \left[\frac{d}{dz} \frac{1}{\left(\frac{d}{dz} f(z) \right)} \cdot \left[\frac{d}{dz} \frac{1}{\left(\frac{d}{dz} f(z) \right)} \right] \right] \right] \right] \quad (15)$$

having $n - 1$ derivatives acting on the right side in addition to the bracketed derivatives acting on $f(z)$ alone. We can rearrange the brackets yielding

$$\frac{d^n}{du^n} g(u) = \left[\frac{1}{\frac{d}{dz} f(z)} \cdot \frac{d}{dz} \right]^{n-1} \frac{1}{\left(\frac{d}{dz} f(z) \right)} \quad (16)$$

The multiplying factor is a differential operator acting on all terms to the right containing any dependence on z . Placing this result to equation (10) yields the simplified inversion formula

$$z = z_0 + \sum_{n=1}^{\infty} \frac{(u - u_0)^n}{n!} \left[\left[\frac{1}{\frac{df(z)}{dz}} \cdot \frac{d}{dz} \right]^{n-1} \frac{1}{\left(\frac{df(z)}{dz} \right)} \right]_{z_0} \quad (17)$$

The necessary, but not sufficient, condition for the new inversion formula to converge is that the first derivative of $f(z)$ must be nonzero at z_0 . The radius of convergence R_1 must be evaluated for each resulting series. If a singularity would appear at z_0 , a translation to a nearby point should be made.

3 The Remainder Term

The remainder term expresses the error caused by breaking the series at N'th term. Thus it is important when broken series are used for approximations and the resulting error must be evaluated. In the following it is derived.

Assuming $f(z)$ is analytic (referring to equation (17)) and by using equation (16) it is straightforward to prove by induction the following expression

$$z = z_0 + \sum_{n=1}^N \frac{(u - u_0)^n}{n!} \left[\left(\frac{1}{f'(z)} \frac{d}{dz} \right)^{n-1} \frac{1}{f'(z)} \right]_{z_0} + \frac{1}{N!} \int_{u_0}^u dt \cdot (u - t)^N \left[\left(\frac{1}{f'(z)} \frac{d}{dz} \right)^N \frac{1}{f'(z)} \right]_{z=g(t), t=f(z)} \quad (18)$$

Here $z = g(u)$ is the inverse function of $u = f(z)$.

3.1 Real Case

For the real variable case, by using the mean value theorem and integrating, we get

$$R_N(u) = \frac{(u - u_0)^{N+1}}{(N + 1)!} \left[\left(\frac{1}{\frac{df(z)}{dz}} \frac{d}{dz} \right)^{N+1} \frac{1}{f'(z)} \right]_{u=f(\zeta), z=g(u)=\zeta} \quad (19)$$

Here ζ is an intermediate value between u_0 and u . This result is analogous to the Lagrange remainder for real functions.

3.2 Complex Case

For the complex variable case we obtain by using the corresponding mean value theorem IV by Curtiss [9]

$$\int_a^z dw f(w) \phi(w) = f(a) \int_a^{a+\theta(z-a)} dw \phi(w) \quad (20)$$

After integration we arrive at

$$R_N(u) = \frac{(1 - (1 - \theta)^{N+1})[(u - u_0)]^{N+1}}{(N + 1)!} \left[\left(\frac{1}{\frac{df}{dz}} \frac{d}{dz} \right)^N \frac{1}{f'(z)} \right]_{z_0} \quad (21)$$

Here θ has a value according to

$$|\theta - 1| < 1 \quad (22)$$

4 Conclusions

The equation (17) represents a simple alternative to the Lagrange-Bürmann formula (equation (6)). The Lagrange-Bürmann formula requires taking limits and repeated use of L'Hospital's rule to remove the singularity. The new formula requires only elementary differentiation and evaluation at z_0 .

Comparison of coefficients in each term between the two formulas is not possible since the expansions are based on polynomials of u . A special case appears when $u_0 = 0$ making the expansions powers of u . This leads to equalities but not directly. One has to approach the limit ($z \rightarrow 0$) in equation (6) finally reaching terms identical with equation (17). Working in the opposite way is not possible.

In spite of its simplicity, this inversion formula can be applied generally. It can be used for inversion of functions and polynomials and for reversion of series. It is valid also for real variables. It is useful for estimating the behavior of the inverse function at some point with a few beginning terms. The radius of convergence needs to be studied for each new series.

The remainder terms both for real variable and complex variable cases have been developed. When estimating the behavior of the inverse function at some point with a few beginning terms, one would need the remainder term for estimating the resulting error.

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