Quest Journals
Journal of Research in Applied Mathematics
Volume 10 ~ Issue 2 (2024) pp: 01-04
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Review Paper

Generating Function For the Riemann Zeta Polynomial On a Limited Range on the Complex Plane Including the Critical Line

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ABSTRACT: *The Riemann Zeta functional polynomial in the complex plane around the critical line needs a generating function. The series representations for instance of type Havil and Hasse are rather complicated as a starting point making the development very difficult. The Dirichlet form is feasible while applying the selected method. The generating function for the polynomial is derived based on it, valid within a limited range covering the critical line. The method published earlier for creating new generating functions makes the work very straightforward.*
KEYWORDS: *Generating functions for functional polynomials, Riemann Zeta function, generating functions, Dirichlet polynomials. Mathematics Subject Classification: 11M06, 05A15. *The author is grateful to Visilab Signal Technologies for supporting this work. Visilab Report #2024-02*

*Received 09 Feb., 2024; Revised 22 Feb., 2024; Accepted 24 Feb., 2024 © The author(s) 2024. Published with open access at **www.questjournals.org***

I. INTRODUCTION

The Riemann Zeta function over the complex plane, especially along the critical line of nontrivial zeros, has been in the focus of strong interest for decades. The Zeta function is usually defined as an infinite series valid over the full complex plane. The well-known expressions below are presented for reference. The Havil series is

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{(k+1)^s} \tag{1}$$

with $s \in C$. The old Hasse format is the following, [2]

$$\zeta(s) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{(k+1)^{(s-1)}} \tag{2}$$

with $s \in C$. The Dirichlet Eta function is

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \tag{3}$$

with $s \in C$, $0 < Re(s) < 1$. It is a result of analytic continuation of the traditional Riemann Zeta function which is valid only in $s \in C$, $Re(s) > 1$. The Eta function is equivalent to the Riemann Zeta function within the range indicated.

Recently some interest has risen into polynomials for the generating function of the Riemann Zeta expressed as a polynomial. The classic literature, for instance [1], [2], [3], [4], ignores this subject totally. Some recent articles [5], [6], [7], [8], [9], [12], indicate a growing interest in the subject of developing a generating function for the Riemann Zeta functional polynomial.

Earlier, a general method has been published for writing a generating function for a regular polynomial [10]. The Zeta function is actually a functional polynomial of exponentials. Recently, a treatment of the simpler case of the Riemann Zeta polynomial's generating function with $Re(s) > 1$, the classic case, was published [11]. That will serve as a short introduction to the parallel method used in this paper.

Since the first two forms above are very complicated and do not seem to be good starting points for the method used, one must use the simpler Dirichlet Eta form. That is sufficient for the purposes of this paper since it will entirely surround the argument range of interest on the complex plane. In the following is developed the generating function for the Dirichlet Eta function valid in a range including entirely the critical line. All formal proofs are ignored to enhance readability.

II. DERIVATION OF THE GENERATING FUNCTION

One can start from (3) by expressing it as a polynomial

$$\zeta_j(s) = \frac{1}{1-2^{1-s}} \sum_{k=1}^{j} \frac{(-1)^{k-1}}{k^s} \tag{4}$$

with $s \in C$, $0 < Re(s) < 1$. One can use the properties of Dirichlet Eta $\zeta(s)$ in the following. At the limit of infinity of the index j, the polynomial will become

$$\zeta_\infty(s) = \zeta(s) \tag{5}$$

and will exist. The lower end of the index is 1. The difference of the polynomial is

$$\Delta\zeta_j(s) = \frac{1}{(1-2^{1-s})} \frac{(-1)^j}{(j+1)^s} \tag{6}$$

Both sides of this expression can be multiplied by

$$\frac{1}{(j+1)^t}$$

and summed over [1,N] to get

$$\sum_{j=1}^{N} \frac{\Delta\zeta_j(s)}{(j+1)^t} = \frac{1}{1-2^{1-s}} \sum_{j=1}^{N} \frac{(-1)^j}{(j+1)^{t+s}} \tag{7}$$

with

$$t \in C, 0 < Re(t) < 1 \tag{8}$$

However, since the range will become challenged in the Dirichlet Eta form of Zeta function, one must apply the following ranges

$$s, t \in C \tag{9}$$

$$0 < Re(s) < 1 \tag{10}$$

$$0 < Re(t) < 1 \tag{11}$$

$$0 < Re(s) + Re(t) < 1 \tag{12}$$

On the right side of (7) the index can be changed a little to obtain

$$\frac{((1 - 2^{1-(s+t)})\zeta_{N+1}(s+t) - 1)}{1 - 2^{1-s}} \tag{13}$$

The known result for partial summation

$$\sum_{k=n_0}^{N} y_k \Delta z_k = |_{n_0}^{N+1} y_k z_k - \sum_{k=n_0}^{N} z_{k+1} \Delta y_k \tag{14}$$

is used in the following to develop the left side of (7) further.

$$\sum_{j=1}^{N} \frac{\Delta\zeta_j(s)}{(j+1)^t} = |_1^{N+1} \frac{\zeta_j(s)}{(j+1)^t} - \sum_{j=1}^{N} \zeta_{j+1}(s) \Delta \frac{1}{(j+1)^t} \tag{15}$$

This is opened up as

$$\sum_{j=1}^{N} \frac{\Delta\zeta_j(s)}{(j+1)^t} = \zeta_{N+1}(s)0 - \zeta_1(s)\frac{1}{2^t} - \sum_{j=1}^{N} \zeta_{j+1}(s)(\frac{1}{(j+2)^t} - \frac{1}{(j+1)^t}) \tag{16}$$

Combining all terms and letting N $\rightarrow \infty$, leads to the final form of the generating function

$$\sum_{j=1}^{\infty} \zeta_j(s)(\frac{1}{j^t} - \frac{1}{(j+1)^t}) = \frac{\zeta(s+t)(1 - 2^{1-(s+t)})}{1 - 2^{1-s}} \tag{17}$$

This result is valid for s, t $\in$ C with

0 < Re(s) < 1
0 < Re(t) < 1
0 < Re(s) + Re(t) < 1

This is an accurate expression within the range indicated, not an approximation.

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III. DISCUSSION

It seems that there are a few attempts to write down a generating function for the Riemann Zeta functional polynomial valid in various regions on the complex plane, [5], [6], [7], [8], [9], [11], [12]. In this paper it has been derived from the Dirichlet Eta function, equation (3) to be valid around the critical line. The polynomial is specified in equation (4). The derivation used a method of treating the difference of the polynomial as was done earlier with regular polynomials [10] and with the classic Riemann Zeta [11], offering a very simple way of processing polynomials. The resulting form (17) is very simple. Since the polynomial is a functional one, the resulting equation does not have a power multiplier on the left side, but a slightly more complicated factor.

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