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Review Paper

On the Generating Function of the Riemann Zeta Polynomial For Positive Real Part Arguments

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ABSTRACT: *Deriving a generating function for the Riemann Zeta functional polynomial appears to be missing as there is no obvious way to proceed. Here the argument of the Zeta function is limited to Re(s) > 1.0 to make sure that the domain excludes the singularity at s = 1. A generating function for the polynomial is derived and is given in a rather simple form for further analysis.*

KEYWORDS: *Generating functions for functional polynomials, Riemann Zeta function, generating functions, Dirichlet polynomials*

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I. INTRODUCTION

The Riemann Zeta function is studied in great width for over 150 years. Mostly the topics have been around the locations and number of nontrivial zeros of it. The Zeta function is usually defined as an infinite series

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \qquad (1)$$

with s in C, Re(s) > 1.0. Recently also some interest has risen of polynomials made of its broken series. However, the classic literature, for instance [1], [2], [3], have no mention of this subject. Some recent articles [4], [5], [6], [7], indicate a growing interest in the subject of developing a generating function for the Riemann Zeta functional polynomial. In the following is developed the generating function valid within the range indicated above. Any formal proofs are left out to enhance readability.

II. DERIVATION OF THE GENERATING FUNCTION

The Riemann Zeta polynomial is not a true polynomial but a functional polynomial consisting of exponentials. It is defined here as follows with the argument range of s in C, Re(s) > 1.0

$$\zeta_j(s) = \sum_{k=1}^{j} \frac{1}{k^s} \qquad (2)$$

This belongs to the set of Dirichlet polynomials

$$D_j(s) = \sum_{k=1}^{j} \frac{a_k}{k^s} \qquad (3)$$

Earlier, a general method has been published for writing a generating function for a regular polynomial [8]. In the following, an analogous method is used in order to progress to results. One already knows a lot of the properties of the ζ(s) and those will be used implicitly in the following. When the index j grows to infinity, the polynomial becomes as

$$\zeta_\infty(s) = \zeta(s) \qquad (4)$$

and will exist. The lower end of the index is 1. The polynomial is always finite when Re(s) > 1.0. The difference of the polynomial is

$$\Delta\zeta_j(s) = \frac{1}{(j+1)^s} \qquad (5)$$

Both sides of this expression can be multiplied by

$$\frac{1}{(j+1)^t}$$

and summed

$$\sum_{j=1}^{N} \frac{\Delta\zeta_j(s)}{(j+1)^t} = \sum_{j=1}^{N} \frac{1}{(j+1)^{t+s}} \qquad (6)$$

with

$$t \in C, Re(t) > 1.0 \qquad (7)$$

One can let N to go to infinity and then use the known result for partial summation

$$\sum_{k=n_0}^{N} y_k \Delta z_k = |_{n_0}^{N+1} y_k z_k - \sum_{k=n_0}^{N} z_{k+1} \Delta y_k \qquad (8)$$

to develop the left side of (6) further.

$$\sum_{j=1}^{\infty} \frac{\Delta\zeta_j(s)}{(j+1)^t} = |_1^\infty \frac{\zeta_j(s)}{(j+1)^t} - \sum_{j=1}^{\infty} \zeta_{j+1}(s) \Delta \frac{1}{(j+1)^t} \qquad (9)$$

This is opened up as

$$\sum_{j=1}^{\infty} \frac{\Delta\zeta_j(s)}{(j+1)^t} = \zeta_\infty(s)0 - \zeta_1(s)\frac{1}{2^t} - \sum_{j=1}^{\infty} \zeta_{j+1}(s)(\frac{1}{(j+2)^t} - \frac{1}{(j+1)^t}) \qquad (10)$$

On the right side of (6) the index can be changed a little to obtain

$$\zeta(s+t) - 1 \qquad (11)$$

Combining all terms leads to

$$-\frac{1}{2^t} - \sum_{j=1}^{\infty} \zeta_{j+1}(s)(\frac{1}{(j+2)^t} - \frac{1}{(j+1)^t}) = \zeta(s+t) - 1 \qquad (12)$$

Change of index on the left side gives

$$\sum_{j=2}^{\infty} \zeta_j(s)(\frac{1}{(j+1)^t} - \frac{1}{j^t}) = -\frac{1}{2^t} - \zeta(s+t) + 1 \qquad (13)$$

One can now complete the sum to a meaningful entity

$$\sum_{j=1}^{\infty} \zeta_j(s)(\frac{1}{(j+1)^t} - \frac{1}{j^t}) = \zeta_1(s)(\frac{1}{2^t} - 1) - \frac{1}{2^t} - \zeta(s+t) + 1 \qquad (14)$$

Thus the final form for the generating function is

$$\sum_{j=1}^{\infty} \zeta_j(s)[\frac{1}{j^t} - \frac{1}{(j+1)^t}] = \zeta(s+t) \qquad (15)$$

This result is valid for Re(s) > 1.0, Re(t) > 1.0. This is an accurate expression, not an approximation.

III. CONCLUSION

It seems that there are a few attempts to write down a generating function for the Riemann Zeta functional polynomial, [4], [5], [6], [7]. In this paper it has been derived and shown in a compact form. The derivation used a straightforward method of treating the difference of the polynomial as was done earlier with regular polynomials [8]. The argument of the Zeta function is limited to Re(s) > 1.0, Re(t) > 1.0 to make sure the domain is outside of the singularity at s + t = 1.

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